SOME SOLUTIONS OF THE PROBLEM OF MOTION OF A BODY WITH A FIXED POINT

(NEKOTORYE RESHENIIA ZADACHI O DVIZHENII TELA, IMEIUSHCHEGO ZAKREPLENNUIU TOCHKU)

PMM Vol.29, № 4, 1965, pp.733-737

E.I. KHARLAMOVA
(Novosibirsk)

(Received March 23, 1965)

The problem of motion of a solid body with a fixed point in a uniform field of gravitational force, as is well known, is reduced to integration of the following system of equations:

$$A\frac{dp}{dt} = (B - C)\,qr + (e_2\gamma_3 - e_3\gamma_2)\,\Gamma \qquad \begin{pmatrix} 123 \\ ABC \\ pqr \end{pmatrix} \qquad (0.1)$$

$$\frac{d\gamma_1}{dt} = r\gamma_2 - q\gamma_3 \qquad (0.2)$$

where e_1, e_2 and e_3 are unit vectors directed from the fixed point to the center of masses of the body, Γ is the product of mass of the body and the distance between the center of masses and the point of support. The remaining designations are conventional [1]; symbols (123, ABC, pqr) designate cyclic permutations. For certain restrictions placed on parameters characterizing the distribution of masses and initial conditions, separate particular solutions of these equations are found.

In recent time various generalizations of the indicated problem which are obtained by complicating of forces acting on the body, are studied intensively. Following Zhukovskii [2],gyroscopic forces are introduced; instead of a uniform force field a central force field (*) is considered, etc. In this connection it became clear that some of the particular solutions obtained for problem (0.1), (0.2) do not have "stability" with respect to generalizations of this kind. In particular, the S.V. Kowalewski's solution does not have a corresponding analog there. In connection with this it is of interest to show solutions of Equations

$$A\frac{dp}{dt} = (B - C)(qr - \mu^2\gamma_2\gamma_3) + \lambda_2 r - \lambda_3 q + (e_2\gamma_3 - e_3\gamma_2)\,\Gamma \qquad (0.3)$$

$$\frac{d\gamma_1}{dt} = r\gamma_2 - q\gamma_3 \qquad (123,\ ABC,\ pqr)$$

which generalize the corresponding solutions of Equations (0.1) and (0.2) (**). Up to the present time such solutions were found only under the condition that at least one of the quantities $\lambda_1^2 + \lambda_2^2 + \lambda_3^2$ or μ is equal to

*) Force function for this case cf., for example, in monograph [1].

**) Solutions given in Sections 4 and 5 was presented in the author's paper sent to *PMM* Feb.24, 1964. This paper was subsequently combined with the present paper.

zero.

Here five solutions of Equations (0.3) and (0.2) are shown, found under the condition $(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)\mu \neq 0$. If this expression is transformed to zero, the indicated solutions either reduce to known particular solutions, or to some generalizations of these solutions.

1. Three integrals of Equations (0.2) and (0.3) are known

$$Ap^2 + Bq^2 + Cr^2 + \mu^2(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2) - 2\Gamma(e_1\gamma_1 + e_2\gamma_2 + e_3\gamma_3) = 2E \qquad (1.1)$$

$$(Ap + \lambda_1)\gamma_1 + (Bq + \lambda_2)\gamma_2 + (Cr + \lambda_3)\gamma_3 = k \qquad (1.2)$$

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 \qquad (1.3)$$

and, as follows from the theory of the last multiplier, it is sufficient to find the fourth integral, not explicitly depending on t and containing an arbitrary constant, in order to reduce the problem to quadratures.

1. A trivial generalization of Lagrange's solution is obtained for conditions

$$B = C, \qquad \lambda_2 = \lambda_3 = 0, \qquad e_2 = e_3 = 0$$

The fourth integral in this case is $p = \text{const}$.

Reduction to quadratures is carried out here by the usual method for Lagrange's case.

2. For conditions $e_1 = 0,\ \lambda_2 = \lambda_3 = 0$ the following particular solution exists

$$p = \frac{d\varphi}{dt}, \qquad q = r = 0, \qquad \gamma_1 = 0, \qquad \gamma_2 = \cos\varphi, \qquad \gamma_3 = \sin\varphi$$

$$A\frac{d^2\varphi}{dt^2} = -\frac{1}{2}\mu^2(B - C)\sin 2\varphi + \Gamma\sin(\varphi - \alpha) \qquad (\tan\alpha = e_3/e_2)$$

which is the trivial generalization of the motion of the physical pendulum applied to the problem under consideration.

3. Motions for which the vector of angular velocity is not changed with respect to the body are found by the usual method [3]. Possible axes of such motion are elements of a cone which yields the following curve in the intersection with the unit sphere (1.3):

$$[(B - C)e_1\gamma_2\gamma_3 + (C - A)e_2\gamma_3\gamma_1 + (A - B)e_3\gamma_1\gamma_2]^2\Gamma +$$
$$+ [(B - C)\lambda_1\gamma_2\gamma_3 + (C - A)\lambda_2\gamma_3\gamma_1 + (A - B)\lambda_3\gamma_1\gamma_2][(\lambda_2e_3 - \lambda_3e_2)\gamma_1 + (\lambda_3e_1 - \lambda_1e_3)\gamma_2 +$$
$$+ (\lambda_1e_2 - \lambda_2e_1)\gamma_3] = \mu^2\Gamma^{-1}[(B - C)\lambda_1\gamma_2\gamma_3 + (C - A)\lambda_2\gamma_3\gamma_1 + (A - B)\lambda_3\gamma_1\gamma_2]^2$$

To each element of this cone corresponds a definite value of angular velocity of uniform rotation

$$\omega = -\Gamma\frac{(B - C)e_1\gamma_2\gamma_3 + (C - A)e_2\gamma_3\gamma_1 + (A - B)e_3\gamma_1\gamma_2}{(B - C)\lambda_1\gamma_2\gamma_3 + (C - A)\lambda_2\gamma_3\gamma_1 + (A - B)\lambda_3\gamma_1\gamma_2}$$

2. If the body has cavities filled with fluid then the condition

$$A = B + C \qquad (2.1)$$

can be applicable to changed moments of inertia entering into equation (0.3) [4]. In the case of a body which does not have such filling, condition (2.1) is fulfilled for an infinitely thin plate. In addition to (2.1) let the parameters be constrained by the following conditions

$$e_1 = 0, \qquad \lambda_1 = 0 \qquad (B^2\lambda_2^2 + C^2\lambda_3^2)\mu^2 = (B^2e_2^2 + C^2e_3^2)\Gamma^2$$

Introducing a new parameter ν , the last condition is presented in the form

$$\mu\lambda_2B = (Be_2\cos 2\nu + Ce_3\sin 2\nu)\Gamma, \qquad \mu\lambda_3C = (Be_2\sin 2\nu - Ce_3\cos 2\nu)\Gamma$$

Under these conditions Equations (0.3) and (0.2) have a solution with three line integrals

$$(B + C)\, p = \mu\, \{(B - C)\, \gamma_1 \cos 2\nu + s\} \tag{2.2}$$
$$q = \mu\, (\gamma_2 \cos 2\nu + \gamma_3 \sin 2\nu) + \lambda_2 / C, \qquad r = \mu\, (\gamma_2 \sin 2\nu - \gamma_3 \cos 2\nu) + \lambda_3 / B$$

Here s is an arbitrary constant.

Introducing (2.2) into integrals (1.1) and (1.2) we arrive at the relationship

$$\mu\, \{(C\gamma_2^2 - B\gamma_3^2) \cos 2\nu + (B + C)\, \gamma_2\gamma_3 \sin 2\nu\} + (B + C) \left(s\gamma_1 + \frac{\lambda_2}{C}\, \gamma_2 + \frac{\lambda_3}{B}\, \gamma_3 \right) = h^*$$

Here h^* is the constant of integration.

It is convenient to study the found solution in a system of coordinates which is rotated with respect to major axes around the first axis by the angle ν in the direction from the second axis to the third (variables and parameters which changed upon rotation are designated by primes).

$$(B' + C')\, p = \mu\, \{(B' - C')\, \gamma_1 + s\}, \qquad q' = \mu\gamma_2' + s_*, \qquad r' = -\mu\gamma_3' + s^* \tag{2.3}$$

$$C' \left(\gamma_2' + \frac{B' + C'}{2\mu C'}\, s_* \right)^2 - B' \left(\gamma_3' - \frac{B' + C'}{2\mu B'}\, s^* \right)^2 = h - s\gamma_1 \tag{2.4}$$

$$\gamma_1^2 + \gamma_2'^2 + \gamma_3'^2 = 1 \tag{2.5}$$

Here h is a new arbitrary constant and

$$s_* = 2\, \frac{2B'\lambda_2' - \lambda_3' (B' - C') \tan 2\nu}{4B'C' - (B' - C')^2 \tan^2 2\nu}, \qquad s^* = 2\, \frac{2C'\lambda_3' - \lambda_2' (B' - C') \tan 2\nu}{4B'C' - (B' - C')^2 \tan^2 2\nu}$$

Introducing (2.3) into Equation (0.2) which in the new axes has the form

$$\frac{d\gamma_1}{dt} = r'\gamma_2' - q'\gamma_3'$$

where

$$\frac{d\gamma_1}{dt} = -2\mu\gamma_2'\gamma_3' + s^*\gamma_2' - s_*\gamma_3' \tag{2.6}$$

Substitution of γ_2' and γ_3' found from (2.4) and (2.5) as a function of γ_1 into this equation permits to establish the dependence of γ_1 (and along with this also of the remaining variables) on time. However, in the general case, for determination of functions $\gamma_2'(\gamma_1)$ and $\gamma_3'(\gamma_1)$ from (2.4) and (2.5) it is necessary to solve the complete algebraic equation of fourth degree. The problem is simplified if $s = 0$ or $\Gamma = 0$. In case $s = 0$, satisfying Equation (2.4), we express γ_2' and γ_3' through a new variable σ

$$\gamma_2' = \left(\frac{h}{C'}\right)^{1/2} \cosh \sigma - \frac{B' + C'}{2\mu C'}\, s_*, \qquad \gamma_3' = \left(\frac{h}{B'}\right)^{1/2} \sinh \sigma + \frac{B' + C'}{2\mu B'}\, s^* \tag{2.7}$$

and consequently

$$p = \mu\, \frac{B' - C'}{B' + C'}\, \gamma_1 = \mu\, \frac{B' - C'}{B' + C'} \left\{1 - \left[\left(\frac{h}{C'}\right)^{1/2} \cosh \sigma - \frac{B' + C'}{2\mu C'}\, s_* \right]^2 - \right.$$
$$\left. - \left[\left(\frac{h}{B'}\right)^{1/2} \sinh \sigma + \frac{B' + C'}{2\mu B'}\, s^* \right]^2 \right\}^{1/2} \tag{2.8}$$
$$q' = \mu \left(\frac{h}{C'}\right)^{1/2} \cosh \sigma - \frac{B' - C'}{2C'}\, s^*, \qquad r' = -\mu \left(\frac{h}{B'}\right)^{1/2} \sinh \sigma + \frac{B' - C'}{2B'}\, s^*$$

In order to determine the dependence of σ on t, it is sufficient to substitute (2.7) and (2.8) into (2.6).

The case $\Gamma = 0$ is discussed in the following Section.

3. In the case

$$\Gamma = 0 \tag{3.1}$$

we have

$$\lambda_1 = \lambda_2 = \lambda_3 = 0 \tag{3.2}$$

and Equations (0.3) take the form

$$A\, \frac{dp}{dt} = (B - C)\, (qr - \mu^2\gamma_2\gamma_3) \qquad (123\ ABC,\ pqr) \tag{3.3}$$

System (3.3), (0.2) was studied by many authors. Apparently the first result here belongs to Clebsch [5]. Occupying himself with the problem of motion of a solid body by inertia in a boundless ideal fluid, Clebsch examined the case when the kinetic energy of the system has the form

$$2T = Ap^2 + Bq^2 + Cr^2 + LR_1^2 + MR_2^2 + NR_3^2$$

(here R_1, R_2 and R_3 are components of impulsive force) and consequently the motion of the boby is described by Equations

$$A\frac{dp}{dt} = (B - C)\,qr + (N - M)\,R_2R_3, \qquad \frac{dR_1}{dt} = rR_2 - qR_3 \quad (123,\ ABC,\ LMN,\ pqr)$$

He established that for condition

$$A\,(M - N) + B\,(N - L) + C\,(L - M) = 0 \tag{3.4}$$

this problem has a complete solution because the fourth integral is determined as

$$Ap^2 + Bq^2 + Cr^2 - \frac{BC}{A}LR_1^2 - \frac{CA}{B}MR_2^2 - \frac{AB}{C}NR_3^2 = \text{const}$$

which permits to find the quadratures.

Apparently, Equation (3.3) follows from equations of Clebsch's problem if one sets in the latter $L = \mu^2A$, $M = \mu^2B$ and $N = \mu^2C$ (condition (3.4) is satisfied). In this connection the fourth integral takes the form

$$A^2p^2 + B^2q^2 + C^2r^2 - \mu^2\,(BC\gamma_1^2 + CA\gamma_2^2\,AB\gamma_3^2) = \text{const} \tag{3.5}$$

Somewhat later this integral was found by Tisserand [6].

Equation (3.3) and integral (3.5) is sometimes without foundation connected with the name of Brun who arrived at these equations much later than Clebsch while examining a quite instructively formulated problem of dynamics of a solid body [7].

Although quadratures to which Equations (3.3), (0.2) [8 and 9] are reduced, give general analytical solutions of the problem, they turn out to be quite clumsy. Kinematic interpretation of the motion of the body is found for this problem only under the condition that the constant of intergration of areas is equal to zero [9]. Examination of particular cases of this problem is therefore of interest such as for example the solution of Steklov with line integrals [10].

We obtain a simple particular solution of Equations (3.3), (0.2) from the solution indicated in the previous Section for conditions (3.1), (3.2). Referring this solution to new axes introduced in the same Section, we obtain

$$(B' + C')\,p = \mu\,\{(B' - C')\,\gamma_1 + s\}, \qquad q' = \mu\gamma_2', \qquad r' = -\mu\gamma_3'$$
$$(B' + C')\,\gamma_2'^2 = B' + h - s\gamma_1 - B'\gamma_1^2, \qquad (B' + C')\,\gamma_3'^2 = C' - h + s\gamma_1 - C'\gamma_1^2$$
$$\frac{d\gamma_1}{dt} = -2\mu\gamma_2'\gamma_3'$$

In this case γ_1, together with the other variables of the problem, are elliptic functions of time.

Apparently, the solution shown here is also a solution of the problem of Clebsch.

The particular case of this solution obtained for $s = 0$ and $\nu = \frac{1}{4}\pi$ was examined by Arkhangel'skii [11]. With new axes the condition $\nu = \frac{1}{4}\pi$ corresponds to $B' = C'$.

4. We shall show one more solution which is of interest because of its connection with investigations of Zhukovskii [2] and Volterra [12]. This solution is characterized by the presence of three relationships

$$\gamma_1 = n_1p + m_1 \qquad (pqr,\ 123) \tag{4.1}$$

We substitute (4.1) into (0.3), (0.2)

$$A\frac{dp}{dt}=(B-C)(1-\mu^2 n_2 n_3)qr+[\lambda_2+e_2 n_3\Gamma-\mu^2(B-C)n_3 m_2]r-$$
$$-[\lambda_3+e_3 n_2\Gamma-\mu^2(C-B)n_2 m_3]q+e_2 m_3\Gamma-e_3 m_2\Gamma-\mu^2(B-C)m_2 m_3 \qquad (4.2)$$
$$n_1\frac{dp}{dt}=(n_2-n_3)qr+m_2 r-m_3 q \qquad (ABC,\ pqr,\ 123)$$

These six relationships of determining p, q and r are compatible if

$$(B-C)(1-\mu^2 n_2 n_3)=(n_2-n_3)An_1^{-1},\qquad \lambda_2+e_2 n_3\Gamma-\mu^2(B-C)n_3 m_2=m_2 An_1^{-1}$$
$$\lambda_3+e_3 n_2\Gamma-\mu^2(C-B)n_2 m_3=m_3 An_1^{-1},\qquad e_2 m_3\Gamma-e_3 m_2\Gamma=\mu^2(B-C)m_2 m_3$$
$$(ABC,\ 123)$$

From this we find

$$n_1=\left(\frac{1+As}{(1+Bs)(1+Cs)\mu^2}\right)^{1/2},\qquad m_1=-\frac{se_1\Gamma}{[2+(B+C)s]\mu^2}\qquad \binom{ABC}{123} \qquad (4.3)$$

if parameters of the system are connected by the following relationships:

$$\lambda_1=-\frac{e_1\Gamma(2+As)}{2+(B+C)s}\left(\frac{(1+Bs)(1+Cs)}{(1+As)\mu^2}\right)^{1/2}\ \binom{ABC}{123} \qquad (4.4)$$

Here s is an arbitrary parameter.

5. We designate $1+As$, $1+Bs$ and $1+Cs$ by a, b and c, respectively, and introduce nondimensional variables relating the components of angular velocity to the quantity $s\sqrt{abc/\mu^2}$; the quantities γ_1, γ_2 and γ_3 to s/μ^2, and the variable t to the quantity $(s\sqrt{abc/\mu^2})^{-1}$.

We substitute (4.3), (4.4) into (4.2), (4.1), passing to nondimensional veriables indicated

$$a\frac{dp}{dt}=(b-c)qr+\frac{e_3\Gamma}{a+b}q-\frac{e_2\Gamma}{a+c}r\quad (abc,\ pqr,\ 123) \qquad (5.1)$$

$$\gamma_1=ap-\frac{e_1\Gamma}{b+c}\qquad (abc,\ pqr,\ 123) \qquad (5.2)$$

Equations (5.1) coincide in form with equations of Zhukovskii [2] in the problem of inertial motion of a body with fluid filled cavity. Two integrals are known of Equations (5.1) and (5.2)

$$ap^2+bq^2+cr^2=h \qquad (5.3)$$
$$\left(ap-\frac{e_1\Gamma}{b+c}\right)^2+\left(bq-\frac{e_2\Gamma}{c+a}\right)^2+\left(cr-\frac{e_3\Gamma}{a+b}\right)^2=\frac{\mu^4}{s^2}$$

and, consequently, the dependence of p, q and r on t is determined by quadratures [12],after that the dependence of the angle of nutation and characteristic rotation on t me is established from (5.2). For determination of the precession angle one additional quadrature is required.

Quadratures to which Volterra reduced Equations (5.1) are complicated in the general case. The problem is simplified if one of the quantities e_i is assumed to be equal to zero.

Let $e_1=0$. Eliminating p from (5.3) we obtain

$$\frac{[q-e_2\Gamma/(c+a)(b-a)]^2}{H/b(b-a)}+\frac{[r-e_3\Gamma/(a+b)(c-a)]^2}{H/c(c-a)}=1 \qquad (5.4)$$
$$\left(H=\frac{\mu^4}{s^2}-ah+\frac{ae_2^2\Gamma^2}{(c+a)^2(b-a)}+\frac{ae_3^2\Gamma^2}{(a+b)^2(c-a)}\right)$$

While satisfying relationship (5.4) we introduce a new variable σ such that

$$q=e_2\Gamma/(c+a)(b-a)+\sqrt{H/b(b-a)}\cos\sigma$$
$$r=e_3\Gamma/(a+b)(c-a)+\sqrt{H/c(c-a)}\sin\sigma \qquad (5.5)$$

(if the signs of denominators in (5.4) are different, then instead of trigonometric functions in (5.5) hyperbolic functions of σ appear). From (5.3), (5.5) we have

$$ap^2 = h - be_2^2\Gamma^2 / (c+a)^2 (b-a)^2 - ce_3^2\Gamma^2 / (a+b)^2 (c-a)^2 -$$
$$- \frac{2be_2\Gamma}{(c+a)(b-a)} \left(\frac{H}{b(b-a)}\right)^{1/2} \cos\sigma - \frac{2ce_3\Gamma}{(a+b)(c-a)} \left(\frac{H}{c(c-a)}\right)^{1/2} \sin\sigma - \quad (5.6)$$
$$- \frac{H}{b-a}\cos^2\sigma - \frac{H}{c-a}\sin^2\sigma$$

Substituting (5.5), (5.6) into (5.1) we conclude that σ is an elliptic function of time.

The solution has an even simpler form in the case $e_2 = e_3 = 0$ (and, consequently, $e_1 = 1$). Equations (5.3) give

$$b(b-c)q^2 = Q + 2a\Gamma p(b+c)^{-1} + a(c-a)p^2$$
$$c(b-c)r^2 = R - 2a\Gamma p(b+c)^{-1} - a(b-a)p^2$$

Constants Q and R are introduced instead of h and k^2

$$Q = k^2 - hc - \Gamma^2(b+c)^{-2}, \qquad R = -k^2 + hb + \Gamma^2(b+c)^{-2}$$

The first equation of (5.1) determines p as an elliptic function of time

$$t = \int_{p_0}^{p} \frac{a\sqrt{bc}\,dp}{\sqrt{[Q + 2a\Gamma p(b+c)^{-1} + a(c-a)p^2][R - 2a\Gamma p(b+c)^{-1} - a(b-a)p^2]}}$$

BIBLIOGRAPHY

1. Lur'e, A.I., Analiticheskaia mekhanika (Analitical Mechanics). Fizmatgiz, 1961.
2. Zhukovskii, N.E., O dvizhenii tverdogo tela, imeiushchego polosti, napolnennye odnorodnoi kapel'noi zhidkost'iu (On motion of a solid body having cavities filled with homogeneous liquid). Sobr.soch., Vol.3, ONTI, 1936.
3. Kharlamov, P.V., O ravnomernykh vrashcheniiakh tela, imeiushchego nepodviznuiu tochku (On uniform rotations of a body with a fixed point). *PMM* Vol.29, № 2, 1965.
4. kharlamova, E.I., Odin chastnyi sluchai integriruemosti uravnenii Eilera-Puassona (One particular case of integrability of equations of Euler-Poisson). Dokl.Akad.Nauk SSSR, Vol.125, № 5, 1959.
5. Clebsch, A., Über the bevegung eines Körpers in einer Flüssigkeit. Math. Annln., Vol.3, 1870.
6. Tisserand, M.F., Sur les mouvements relatifs à la surface de la Terre. Comptes Rendus, Vol.75, № 26, 1872.
7. Brun, F., Rotation kring on fix punkt. Ofversigt af Kongl. Svenska Vetensk.Akad.Forhandlinger, Stokholm, 1893.
8. Kobb, C., Sur le problème de la rotation d'un corps autour d'un point fixe. Bull.Soc.math., Vol.23, 1895.
9. Kharlamova, E.I., O dvizhenii tverdogo tela vokrug nepodvizhnoi tochki v tsentral'nom n'iutonovskom pole sil (On motion of a solid body around a fixed point in a central Newtonian field of force). Izv.sib. Otdel.Akad.Nauk SSSR, № 6, 1959.
10. Steklov, V.A., O nekotorykh vozmozhnykh sluchaiakh dvizheniia tverdogo tela v zhidkosti (On some possible cases of motion of a solid body in a liquid). Trudy Otd.fiz.Nauk Obshch.Liub.Estest., Vol.8, № 2, 1896.
11. Arkhangel'skii, Iu.A., Ob odnom dvizhenii diska v n'iutonovskom pole sil (On a motion of a disk in a Newtonian force field). *PMM* Vol.28, № 4, 1964.
12. Volterra, V., Sur la théorie des variations des latitudes. Acta Math., Vol.22, 1899.

Translated by B.D.